A LINEAR ANALYSIS OF STEADY SURFACE WAVES ON A VISCOUS LIQUID FLOWING DOWN AN INCLINED PLANE

by

P. Smith*

1. *Introduction*

The steady flow of a viscous liquid of constant depth down an inclined plane is a well-known simple example of a free surface flow (it is given, for example by Berker [1], p. 16). The simplicity of the solution suggests that is a useful starting point for an investigation of steady viscous waves caused by the introduction of rigid bumps and obstacles into the bed of the stream. It is assumed that the ratio of the maximum disturbance height to the mean depth of liquid is small, and that, as a consequence of this, the linearization of the Navier-Stokes equations and the boundary conditions is justifiable.

The nature of the boundary-value problem suggests that the characteristics of the free surface will depend on three parameters - the angle of inclination β of the plane, the Reynolds number R and the ratio of the liquid depth to a representative disturbance length. For example the free surface of a deep layer of liquid is unlikely to be significantly affected by small variations in the lower boundary. A large Reynolds number could lead to wave breaking, instability and turbulence in a thin film of liquid. The precise interaction between these parameters is not easy to evaluate.

The simplest problem is that of slow motion in which Reynolds number is small and the inertia terms are negligible compared with the viscous ones. This class of flows is solved fully for linearized boundary conditions, the free surface and all flow variables being expressed as Fourier integrals containing an arbitrary perturbation term. This is achieved by standard Fourier transform techniques. The flow over a wavy boundary of long wavelength is looked at in some detail. For a smooth hump in the inclined plane, asymptotic expansions for the free surface can be found for shallow flow. This is essentially a long wave theory, and it does suggest a novel asymptotic method based on an iteration procedure in which small inertia effects can be incorporated.

The stability of surface waves on an inclined viscous flow has been discussed previously by Benjamin [2] (a list of more recent papers on this topic is given by Wehausen [3], p. 575). In Benjamin's work the stability of the free surface is investigated when a sinusoidal perturbation in space and time is imposed on it and allowed to develop. Since instabilities are largely a consequence of inertia effects they are unlikely to appear in the slow motion of a liquid. However the later work indicates how the crest of a single wave grows in height as the Reynolds number is increased. An experimental study of waves on water films has been conducted by Binnie [4], [5]. Experimental confirmation of the results presented here should not be difficult for low Reynolds number. For liquids of low viscosity such as water surface instabilities could prove difficult to eliminate since they can arise at any Reynolds number if they have the appropriate wavelength.

The analysis of boundary-value problems in free surface viscous flow is difficult even with the equations and boundary conditions linearized. This class of steady flows provides probably the simplest type which are capable of detailed analytic treatment.

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* University of Keele, Staffordshire, England.

2. *The basic flow*

The laminar flow of a viscous liquid down an inclined plane supplies a ready-made rectilinear motion which is maintained by a balance between gravity and the viscous drag on the bed of the stream. Take a coordinate system as displayed in figure 1. The flow is entirely two-dimensional with

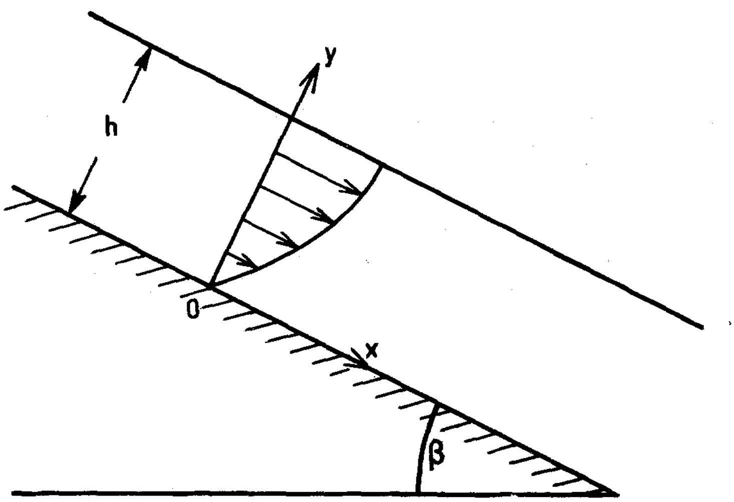

Figure 1. The coordinate scheme.

a parabolic velocity distribution given by

$$u^{(0)} = gy\,(2h - y)\,\sin\beta/2\nu \tag{2.1}$$

where ν is the kinematic viscosity of the incompressible liquid.

Benjamin [2] remarks that the character of any disturbance to the flow depends largely on the Reynolds number defined by $R = P/\nu$, where P is the rate of volume flow per unit span of the stream. Clearly $P = \frac{2}{3}hu^{(0)}(h)$ and $R = h^3 g \sin\beta/3\nu^2$. We are interested in the case of small R.

3. *The linearized equations and boundary conditions*

The two-dimensional Navier-Stokes equations can be written

$$\rho u u_x + \rho v u_y = -p_x + \rho g \sin\beta + \rho\nu\,(u_{xx} + u_{yy}), \tag{3.1}$$

$$\rho u v_x + \rho v v_y = -p_y - \rho g \cos\beta + \rho\nu\,(v_{xx} + v_{yy}), \tag{3.2}$$

where p is the pressure and the two components of the gravitational force have been included. In addition the velocity components u and v are related through the continuity condition

$$u_x + v_y = 0. \tag{3.3}$$

Let the lower boundary of the flow be changed to $y = \eta_1(x)$ where $\eta_1(x)$ is small. We shall say more about comparisons of typical lengths later. Suppose that the consequent free surface perturbation becomes $y = h + \eta_2(x)$ with the tacit assumption that $\eta_2(x)$ is also small. The no-slip condition gives

$$u = v = 0 \tag{3.4}$$

on $y = \eta_1(x)$. The kinematic surface condition becomes

$$u\eta'_2 - v = 0 \tag{3.5}$$

on $y = \eta_2(x)$. The stress must also be continuous across the surface with the result that

$$(-p + 2\rho\nu u_x)\eta'_2 - \rho\nu(u_y + v_x) = 0, \tag{3.6}$$

$$\rho\nu(u_y + v_x)\eta'_2 - (-p + 2\rho\nu v_y) = 0, \tag{3.7}$$

here is has been assumed for convenience that the external pressure is zero. This means that the unperturbed pressure distribution in the liquid has been taken as

$$p^{(0)}(y) = \rho g(h - y)\cos\beta; \tag{3.8}$$

this will be the actual pressure apart from an additive constant. Surface tension is absent. There are five boundary conditions to be satisfied which also contain and determine the unknown function $\eta_2(x)$.

We now adopt a standard perturbation procedure and write

$$u = u^{(0)} + u^{(1)}, \quad v = v^{(1)}, \quad p = p^{(0)} + p^{(1)} \tag{3.9}$$

The substitution of these expressions into the Navier-Stokes equations (3.1) and (3.2) and the rejection of terms of higher degree than the first in the superscript (1) leads to

$$\rho u^{(0)}u^{(1)}_x + \rho v^{(1)}u^{(0)}_y = -p^{(1)}_x + \rho\nu(u^{(1)}_{xx} + u^{(1)}_{yy}), \tag{3.10}$$

$$\rho u^{(0)}v^{(1)}_x = -p^{(1)}_y + \rho\nu(v^{(1)}_{xx} + v^{(1)}_{yy}). \tag{3.11}$$

The equation of continuity remains as

$$u^{(1)}_x + v^{(1)}_y = 0. \tag{3.12}$$

The boundary conditions (3.4) - (3.7) are now completely linearized in terms of $u^{(1)}$, $v^{(1)}$, $p^{(1)}$, η_1 and η_2 and its derivative. With Taylor expansions used where necessary, they become

$$u^{(0)}_y(o)\eta_1(x) + u^{(1)}(x,o) = 0, \tag{3.13}$$

$$v^{(1)}(x,o) = 0, \tag{3.14}$$

$$u^{(0)}(h)\eta'_2(x) - v^{(1)}(x,h) = 0, \tag{3.15}$$

$$u^{(0)}_{yy}(h)\eta_2(x) + u^{(1)}_y(x,h) + v^{(1)}_x(x,h) = 0, \tag{3.16}$$

$$p^{(0)}_y(h)\eta_2(x) + p^{(1)}(x,h) - 2\rho\nu v^{(1)}_y(x,h) = 0. \tag{3.17}$$

The function η_2 can be eliminated between (3.15), (3.16) and (3.17) to leave the two free-surface conditions:

$$u^{(0)}_{yy}(h)v^{(1)}(x,h) + u^{(0)}(h)\left\{u^{(1)}_{yx}(x,h) + v^{(1)}_{xx}(x,h)\right\} = 0, \tag{3.18}$$

$$p^{(0)}_y(h)v^{(1)}(x,h) + u^{(0)}(h)\left\{p^{(1)}_x(x,h) - 2\rho\nu v^{(1)}_{yx}(x,h)\right\} = 0. \tag{3.19}$$

When the boundary-value problem has been solved, η_2 can be found from (3.15).

4. *Stokes flow*

The creeping flow of a liquid down the plane is the simplest case. For small Reynolds number R we can discard the inertia terms on the left-hand side of equations (3.10) and (3.11). The equation of continuity (3.12) implies the existence of a perturbation stream function $\psi^{(1)}$ defined by $u^{(1)} = \psi_y^{(1)}$ and $v^{(1)} = -\psi_x^{(1)}$. This stream function satisfies the familiar biharmonic equation

$$\nabla^4\psi^{(1)} = 0. \tag{4.1}$$

The boundary conditions (3.13), (3.14), (3.18) and (3.19) become

$$\psi_y^{(1)}(x,o) = -2hk\eta_1(x), \quad \psi_x^{(1)}(x,o) = 0; \tag{4.2}$$

$$2\psi_x^{(1)}(x,h) + h^2\psi_{yyx}^{(1)}(x,h) - h^2\psi_{xxx}^{(1)}(x,h) = 0, \tag{4.3}$$

$$2\cot\beta\ \psi_x^{(1)}(x,h) + 3h^2\psi_{xxy}^{(1)}(x,h) + h^2\psi_{yyy}^{(1)}(x,h) = 0, \tag{4.4}$$

where $k = g\sin\beta/2\nu$. In these equations $u^{(0)}$ and $p^{(0)}$, given by (2.1) and (3.8), have been introduced and the pressure $p_x^{(1)}$, given by (3.10), has been substituted into (4.4). Note that if $\eta_1(x)$ is an even function of x then $\psi^{(1)}(x,y)$ will only be even if $\cot\beta = 0$ since the only non-symmetric term occurs in the undisturbed transverse pressure gradient in (4.4). The free surface will only be "in phase" with the disturbing effect for a vertical film of liquid.

The solution of (4.1) is required in the strip $-\infty < x < \infty$, $0 \leq y \leq h$ and this invites the application of Fourier transform methods. Let the Fourier transform of $\psi^{(1)}(x,y)$ with respect to x be

$$\bar{\psi}^{(1)}(\alpha,y) = \int_{-\infty}^{\infty}\psi(x,y)e^{-i\alpha x}\,dx.$$

The transform of the biharmonic equation is

$$\bar{\psi}_{yyyy}^{(1)} - 2\alpha^2\bar{\psi}_{yy}^{(1)} + \alpha^4\bar{\psi}^{(1)} = 0, \tag{4.5}$$

and the transformed end conditions are

$$\bar{\psi}_y^{(1)}(\alpha,o) = -2hk\bar{\eta}_1(\alpha), \quad \bar{\psi}^{(1)}(\alpha,o) = 0; \tag{4.6}$$

$$(2+\alpha^2h^2)\bar{\psi}^{(1)}(\alpha,h) + h^2\bar{\psi}_{yy}^{(1)}(\alpha,h) = 0, \tag{4.7}$$

$$2i\alpha\cot\beta\ \bar{\psi}^{(1)}(\alpha,h) - 3\alpha^2h^2\bar{\psi}_y^{(1)}(\alpha,h) + h^2\bar{\psi}_{yyy}^{(1)}(\alpha,h) = 0, \tag{4.8}$$

where $\bar{\eta}_1(\alpha)$ is the Fourier transform of $\eta_1(x)$. The solution of the system represented by (4.5) - (4.8) is a piece of tedious but routine analysis which gives

$$\begin{aligned}\bar{\psi}^{(1)}(\alpha,y) = 2hk\bar{\eta}_1(\alpha)\Big[\cot\beta\ &\{h\sinh\alpha y - y\sinh\alpha h\cosh\alpha(h-y)\} - \\ -i\alpha h\ &\{(h-y)(1+\alpha^2h^2)\sinh\alpha y + y\alpha h\cosh\alpha(h-y)\cosh\alpha h\}\Big]\Big/ Q(\alpha,h,\beta),\end{aligned} \tag{4.9}$$

where

$$Q(z,\beta) = (\sinh z\cosh z - z)\cot\beta + iz^2(1+z^2+\cosh^2 z). \tag{4.10}$$

Equation (4.9) gives the Fourier transform of the perturbation stream function, (and hence the velocity components and pressure) can be expressed as a Fourier integral by the inversion theorem.

The transform of the free surface can be obtained by taking transforms of (3.15):

$$\bar{\eta}_2(\alpha) = -\bar{\psi}^{(1)}(\alpha, h)/kh^2.$$

The inversion theorem then gives

$$\eta_2(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{\eta}_1(\alpha)e^{i\alpha x}\ G(\alpha h, \beta)d\alpha \tag{4.11}$$

where (4.9) has been substituted and where

$$G(z, \beta) = 2\, iz^2 \cosh z/Q(x, \beta). \tag{4.12}$$

Integral (4.11) cannot be evaluated explicitly in general. However some special solutions can be obtained by inverse methods. For example, in the case $\beta = \frac{1}{2}\pi$, the choice of

$$\bar{\eta}_1(\alpha) = de^{-c|\alpha|}\,(1 + \alpha^2h^2 + \cosh^2\alpha h)$$

leads to

$$\eta_2(x) = \frac{d}{\pi}\left\{\frac{c-h}{(c-h)^2 + x^2} + \frac{c+h}{(c+h)^2 + x^2}\right\},$$

and

$$\eta_1(x) = \frac{d}{\pi}\left[\frac{3c}{2(c^2+c^2)} - ch^2\frac{d^2}{dx^2}\left\{\frac{1}{c^2+x^2}\right\} + \frac{c-2h}{4\{(c-2h)^2 + x^2\}} + \frac{c+2h}{4\{(c+2h)^2 + x^2\}}\right],$$

provided $c > 2h$. The required transforms can be read off from the table of Fourier cosine transforms given by Erdélyi [6].

5. *The stationary sine wave*

The simplest case in Stokes flow occurs with the bed in the shape of a cosine wave in which $\eta_1(x) = d\cos\omega x$. This particular boundary-value problem can be solved quite easily *ab initio*, but since we have already derived the transform of the stream function in (4.9) we can read off the solution by expressing the Fourier transform of $\eta_1(x)$ in terms of generalised functions. If $\delta(x)$ is the Dirac delta function, then (see Jones [7], p. 469)

$$\bar{\eta}_1(\alpha) = \pi d\left\{\delta(\alpha-\omega) + \delta(\alpha+\omega)\right\}$$

with the result that

$$\psi^{(1)}(x, y) = 2khd\,\mathcal{R}e\left\{\left[\cot\beta\left\{h\sinh\omega y - y\sinh\omega h\cosh\omega(h-y)\right\} - i\omega h\left\{(h-y)(1+\omega^2h^2)\sinh\omega y + y\omega h\cosh\omega(h-y)\cosh\omega h\right\}\right]e^{i\omega x}\Big/Q(\omega h, \beta)\right\}. \tag{5.1}$$

It follows from (4.11) that

$$\eta_2(x) = d\,\mathcal{R}e\left\{G(\omega h, \beta)e^{i\omega x}\right\}. \tag{5.2}$$

Put $G(\omega h, \beta) = Ae^{i\gamma}$ so that (5.2) reads as

$$\eta_2(x) = Ad\cos(\omega x + \gamma) \tag{5.3}$$

where

$$A = 2\epsilon^2\cosh\epsilon\Big/\left\{(\epsilon - \sinh\epsilon\cosh\epsilon)^2\cot^2\beta + \epsilon^4(1+\epsilon^2 + \cosh^2\epsilon)^2\right\}^{\frac{1}{2}}, \tag{5.4}$$

$$\tan\gamma = (\sinh\epsilon\cosh\epsilon - \epsilon)\cot\beta/\epsilon^2(1+\epsilon^2+\cosh^2\epsilon), \tag{5.5}$$

and $\epsilon = \omega h$, which is proportional to the ratio of the mean depth of liquid to the wavelength $2\pi/\omega$.

A graph of the surface amplitude ratio A against ϵ is shown in figure 2

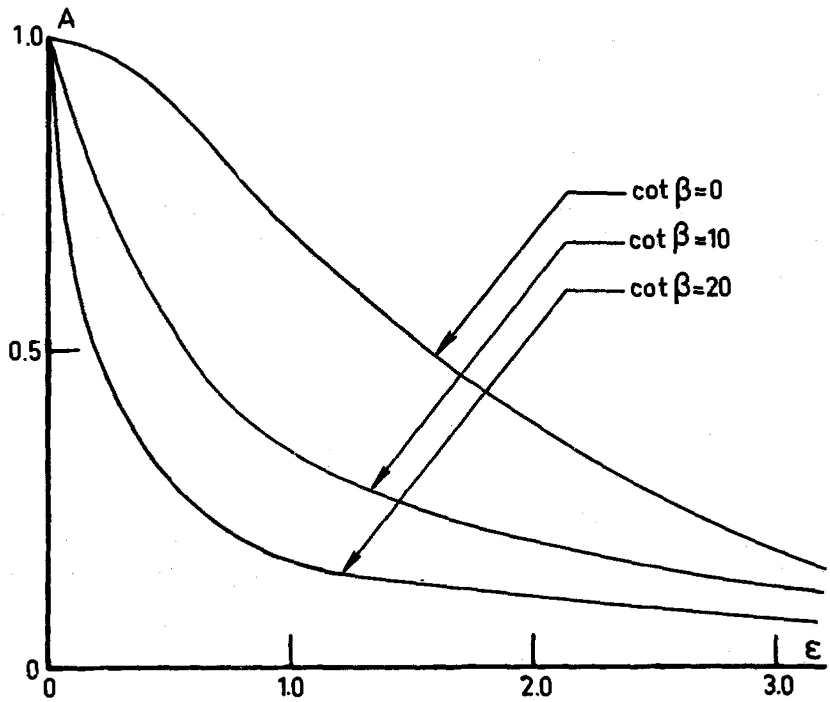

Figure 2. The amplitude ratio A of the surface wave plotted against the depth ratio ϵ for three values of cot β.

for three values of cot β. For any given depth the amplitude decreases as the angle of inclination β decreases, the maximum amplitude occurring for flow down a vertical wall (cot β = 0). For all values of β, the amplitude decreases as ϵ increases. This is to be expected since the surface becomes insensitive to small variations in the stream bed for a large depth of liquid.

The two waves are out of phase except for flow down a vertical plate. The surface wave precedes the forcing wave by a distance γ/ω although with reduced amplitude. A graph of $\tan\gamma\tan\beta$ against ϵ is shown in figure 3. The curve has a maximum at $\epsilon = 1.25$ approximately, which indicates that the greatest difference for any inclination β occurs for this depth-ratio of liquid. Its value is given by $\tan\gamma = 0.19\cot\beta$ from which we see that γ varies between 0 for $\beta = \frac{1}{2}\pi$ and $\frac{1}{2}\pi$ for $\beta = 0$. If the liquid adheres to the underside of and inclined plate, β lies in the range $\frac{1}{2}\pi < \beta < \pi$ and the surface wave *follows* the forcing wave by a distance γ/ω.

We would expect this phase difference between the surface wave and the bed of the stream to lead to separation of the main stream from the boundary in the troughs of the wave. This stretches the linear theory a little but since the results seem plausible, the matter is worth pursuing. The condition for separation is given approximately by **n**. grad u = 0, where **n** is the normal to the boundary and account has been taken of the small slope of the boundary. This represents the point at which the stream-line separates and the normal velocity gradient reverses. Appropriately linearized this condition becomes

$$u_y^{(0)}(o) + u_{yy}^{(0)}(o)\eta_1(x) + u_y^{(1)}(x,o) = 0. \tag{5.6}$$

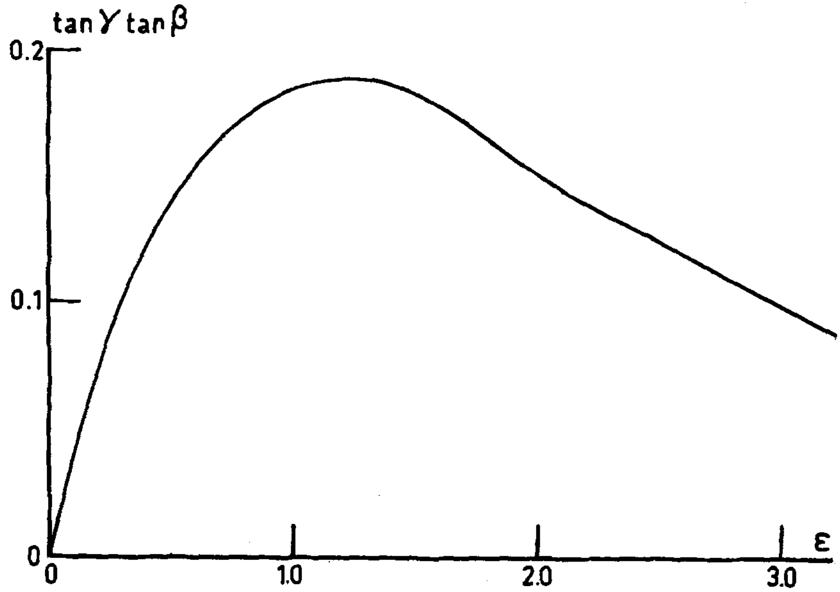

Figure 3. The variation of phase difference λ with depth.

The conclusions can only remain tentative since (5.6) contains a mixture of terms of different orders.

The introduction of the stream function $\psi^{(1)}(x,y)$ from (5.1) into (5.6) leads to

$$\frac{d}{h} = \frac{C^2\cot^2\beta + D^2}{(BC - AD)\cot\beta\sin\omega x - (AC\cot^2\beta + BD)\cos\omega x.} \tag{5.7}$$

where the constants A, B, C, D are given by

$$A = \epsilon\cosh 2\epsilon - \tfrac{1}{2}\sinh 2\epsilon, \quad B = \epsilon^2(\epsilon\sinh 2\epsilon - \cosh^2\epsilon),$$

$$C = \tfrac{1}{2}\sinh 2\epsilon - \epsilon, \quad D = \epsilon^2(1 + \epsilon^2 + \cosh^2\epsilon).$$

The critical depth-ratio $(d/h)_{crit}$ at which separation starts to occur for fixed ϵ and $\cot\beta$ will be given by the envelope of (5.7) with x treated as the parameter. This envelope can be derived in the usual way by eliminating x between (5.7) and the equation obtained by equating to zero the partial derivative of (5.7) with respect to x:

$$\tan\omega x = (AD - BC)\cot\beta/(AC\cot^2\beta + BD). \tag{5.8}$$

The critical depth-ratio which results from the elimination of x between (5.7) and (5.8) is presented in figure 4 for selected values of $\cot\beta$. Remembering that the validity of the approximation depends on d/h being small, we observe that separation will occur for given values of ϵ and $\cot\beta$ if d/h exceeds the critical value indicated by the appropriate curve. For deep flow over a wave of short wavelength the angle of inclination becomes irrelevant. This follows since

$$\left(\frac{d}{h}\right)_{crit} \sim \frac{1}{2\epsilon} \text{ as } \epsilon \to \infty$$

independently of $\cot\beta$. The figure indicates that this holds with reasonable accuracy for $\epsilon > 3$.

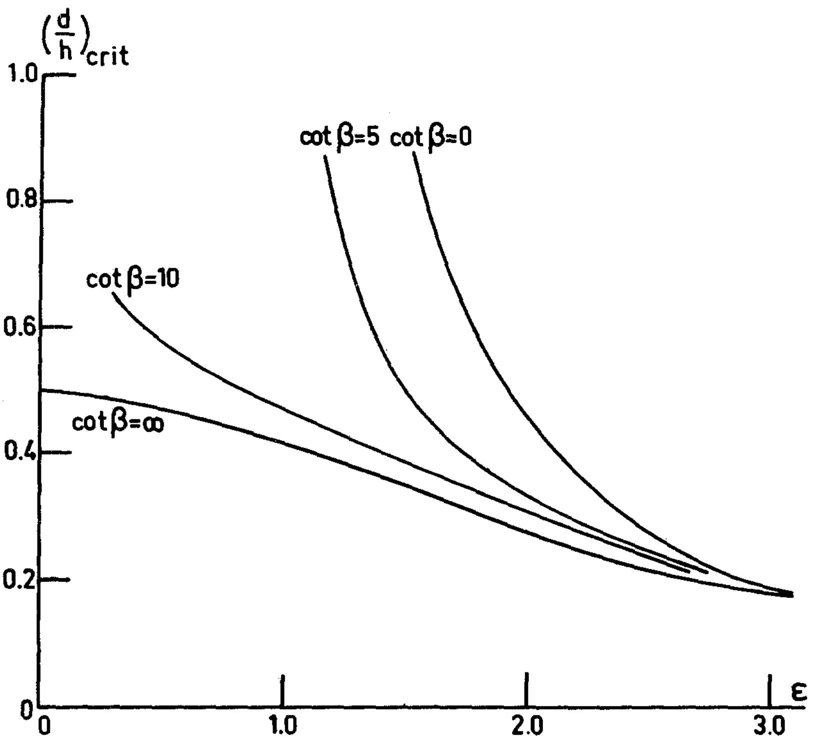

Figure 4. The critical curves for seperation for several values of cot β.

6. *Asymptotic estimates*

We note firstly that discontinuous boundaries can be treated. For example, the finite step represented by

$$\eta_1(x) = d, \quad |x| < a; \quad \eta_1(x) = 0, \quad |x| > a$$

has the Fourier transform $\bar{\eta}_1(\alpha) = 2d \sin \alpha a/\alpha$. The corresponding free surface for laminar flow over this step is, from equation (4.11),

$$\eta_2(x) = \frac{d}{\pi}\int_{-\infty}^{\infty} \frac{\sin \alpha a}{\alpha}\, e^{i\alpha x}\, G(\alpha h, \beta)d\alpha.$$

A few test calculations have shown that the flow displays the same general characteristics as flow over a wavy boundary. The behaviour of G for large $|\alpha|$ also ensures that the free surface is a smooth function of x.

The flow over a single smooth hump is now considered since it suggests generalisations which include small inertia effects. In order to fix ideas we shall concentrate on the particular perturbation given by

$$\eta_1(x) = da^2/(x^2 + a^2),$$

with the understanding that the methods can be used equally successfully for other shapes. The perturbation is symmetric about x = 0 with its maximum height of d there. Its Fourier transform is

$$\bar{\eta}_1(\alpha) = ad\pi e^{-a|\alpha|},$$

and it follows again from (4.11) that

$$\eta_2(x) = \tfrac{1}{2}\, ad \int_{-\infty}^{\infty} e^{-a|\alpha| + i\alpha x} G(\alpha h, \beta)d\alpha,$$

$$= \tfrac{1}{2}\, d\lambda \left\{ \int_0^\infty e^{-(a-ix)z\lambda/a}\, G(z,\beta)dz + \int_0^\infty e^{-(a+ix)z\lambda/a}\, G(-z,\beta)dz \right\} \tag{6.1}$$

where $\lambda = a/h$. Our aim is the construction of an asymptotic expansion of these integrals for large λ, which corresponds to a shallow-liquid approximation. Since $G(z,\beta)$ is a regular function of z in some neighbourhood of the origin in the z-plane, we can write

$$G(z,\beta) = \sum_{n=0}^\infty g_n(\beta) z^n \quad \text{for } |z| \le \delta(\beta) \tag{6.2}$$

for some $\delta(\beta) > 0$. A simple estimate also shows that $|G(z,\beta)|$ decreases exponentially as $\Re(z)\to\pm\infty$. These conditions are sufficient for the application of Watson's lemma (Copson [8], p. 49) to both integrals in (6.1) as $|(a-ix)\lambda/a|\to\infty$ in the first integral and as $|(a+ix)\lambda/a|\to\infty$ in the second. Reading off the expansion, we find that

$$\begin{aligned}\eta_2(x) &\sim \tfrac{1}{2} d\lambda \left[\sum_{n=0}^\infty g_n(\beta) n! \left\{(a-ix)\lambda/a\right\}^{-n-1} + \sum_{n=0}^\infty (-1)^n g_n(\beta) n! \left\{(a+ix)\lambda/a\right\}^{-n-1} \right], \\ &\sim \sum_{n=0}^\infty (-i)^n g_n(\beta)\, \eta_1^{(n)}(x)\, (a/\lambda)^n,\end{aligned} \tag{6.3}$$

as $\lambda\to\infty$ for any positive a and any fixed x. In the last expansion we have deliberately written the terms in derivatives of $\eta_1(x)$ since it suggests a general method of solving the problem asymptotically. We shall develop this is the final section.

The first six coefficients in the power series expansion for $G(z,\beta)$ given by (6.2) are:

$$g_0(\beta) = 1, \quad g_1(\beta) = \tfrac{1}{3} i \cot\beta,$$

$$g_2(\beta) = -(9 + 2\cot^2\beta)/18, \quad g_3(\beta) = -i\cot\beta(117 + 10\cot^2\beta)/270,$$

$$g_4(\beta) = \tfrac{3}{8} - \tfrac{13}{30}\cot^2\beta - \tfrac{1}{81}\cot^4\beta,$$

$$g_5(\beta) = i\cot\beta\left(-\tfrac{3841}{2520} + \tfrac{41}{270}\cot^2\beta - \tfrac{1}{243}\cot^4\beta\right).$$

A comparison of the terms in the expansion indicates that $h\cot\beta$ should not exceed unity for reasonable numerical results not requiring more than the six coefficients given above. The reason, in part, for this restriction lies in the behaviour of $\Delta(\beta)$ in (6.2). As $\cot\beta\to\infty$ we find that $\Delta(\beta)\to 0$. this being caused by a pole of $G(z,\beta)$ which is situated on the imaginary axis in the z - plane. The surface wave in a particular case is shown in figure 5 of the next section. The special example outlined here has its counterparts for humps of other shapes and various asymptotic methods (such as the method of steepest descent) can be used in the shallow-liquid approximation. The asymptotic expansion of the free surface is given by (6.3) provided the Fourier transform is an exponentially decreasing function as $|\alpha|$ increases.

7. *An asymptotic solution containing small inertia affects*

The elimination of the pressure $p^{(1)}$ between (3.10) and (3.11) with the inertia terms retained leads to

$$\nu(u^{(1)}_{xxy} + u^{(1)}_{yyy} - v^{(1)}_{xxx} - v^{(1)}_{yxx}) + u^{(0)} v^{(1)}_{xx} - u^{(0)} u^{(1)}_{xy} - u^{(0)}_{yy} v^{(1)} = 0.$$

In terms of the stream function $\psi^{(1)}$ this equation becomes

$$\begin{aligned}&\psi^{(1)}_{yyyy} + 2\psi^{(1)}_{xxyy} + \psi^{(1)}_{xxxx} - \frac{3R}{2h^3} y(2h-y)\psi^{(1)}_{xxx} - \\ &- \frac{3R}{2h^3} y(2h-y)\psi^{(1)}_{xyy} - \frac{3R}{h^3}\psi^{(1)}_{x} = 0,\end{aligned} \tag{7.1}$$

where $u^{(0)}$ has been introduced from (2.1) and R is the Reynolds number defined in Section 2. Of the boundary conditions, (4.2) and (4.3) are unchanged, but (4.4) is modified by the additional inertia effects in the pressure gradient. Substituting for the pressure gradient $p_x^{(1)}$ from (3.1) into (3.19), we deduce that (4.4) must be replaced by

$$4 \cot \beta \, \psi_x^{(1)}(x,h) + 6h^2\psi_{xxy}^{(1)}(x,h) + 2h^2\psi_{yyy}^{(1)}(x,h) - 3Rh\psi_{xy}^{(1)}(x,h) = 0. \tag{7.2}$$

We now attempt an asymptotic solution

$$\psi^{(1)}(x,y) \sim \sum_{n=0}^{\infty} t_n(y)\eta_1^{(n)}(x), \tag{7.3}$$

where $t_n(y)$ is found by iteration, the successive steps in the iteration being determined by equating like derivatives of $\eta_1(x)$ in equation (7.1) and in the boundary conditions (4.2), (4.3) and (7.2). Justification for this method stems from the asymptotic behaviour of the Fourier integral for slow shallow flow over a hump.

Routine methods give the first three equations

$$t_0'''' = 0,$$

$$t_1'''' - \frac{3R}{2h^3} y(2h-y)t_0'' - \frac{3R}{h} t_0 = 0,$$

$$t_2'''' + 2t_0'' - \frac{3R}{2h^3} y(2h-y)t_1'' - \frac{3R}{h^3} t_1 = 0,$$

subject to the three sets of end conditions

$$t_0(o) = 0, \ t_0'(o) = -2hk, \ 2t_0(h) + h^2t_0''(h) = 0, \ t_0'''(h) = 0;$$

$$t_1(o) = t_1'(o) = 0, \ 2t_1(h) + h^2t_1''(h) = 0,$$

$$4 \cot \beta \, t_0(h) - 3Rht_0'(h) + 2h^2t_1'''(h) = 0;$$

$$t_2(o) = t_2'(o) = 0, \ 2t_2(h) + h^2t_2''(h) - h^2t_0(h) = 0,$$

$$4 \cot \beta \, t_1(h) + 6h^2t_0'(h) + 2h^2t_2'''(h) - 3Rht_1'(h) = 0.$$

All solutions are polynomials in y and, in principle, any number of terms can be obtained. For the system displayed above

$$t_0(y) = ky(y-2h), \quad t_1(y) = \tfrac{1}{3}ky^2(y-2h) \cot \beta,$$

$$t_2(y) = -\tfrac{1}{9}kh^2y^2(9 + 2 \cot^2 \beta) + \tfrac{1}{9}khy^3(6 + \cot^2 \beta) - \tfrac{1}{6}ky^4 + \frac{Rky^2 \cot \beta}{420h^3}(9h^5 - 14h^2y^3 + 7hy^4 - y^5).$$

Note that the Reynolds number does not appear in the first two terms of the expansion. The free surface is now given by the integral of (3.15):

$$\eta_2(x) \sim -\frac{1}{kh^2}\psi^{(1)}(x,h) \sim -\frac{1}{kh^2}\left\{ t_0(h)\eta_1(x) + t_1(h)\eta_1'(x) + t_2(h)\eta_1''(x) \right\}$$

$$\sim \eta_1(x) + \tfrac{1}{3}h \cot \beta \, \eta_1'(x) + h^2(\tfrac{1}{2} + \tfrac{1}{9} \cot^2 \beta - \frac{R}{42} \cot \beta)\eta_1''(x).$$

Some confidence in the method is gained by observing that expansion (6.3) reappears if R is put equal to zero in the expansion above.

We now consider the particular case $\eta_1(x) = da^2/(x^2 + a^2)$ again. Non-dimensionalize x by introducing $X = x/a$, whence

$$\frac{\eta_2(x)}{d} \sim \frac{1}{1 + X^2} - \frac{2X\epsilon \cot \beta}{3(1 + X^2)^2} + (1 + \tfrac{2}{9} \cot^2\beta - \frac{R}{21} \cot \beta)\frac{(3X^2 - 1)\epsilon^2}{9(1 + X^2)^3},$$

where $\epsilon = h/a$.

Figure 5 shows the free surface shape of $\eta_2(x)/d$ plotted against X with

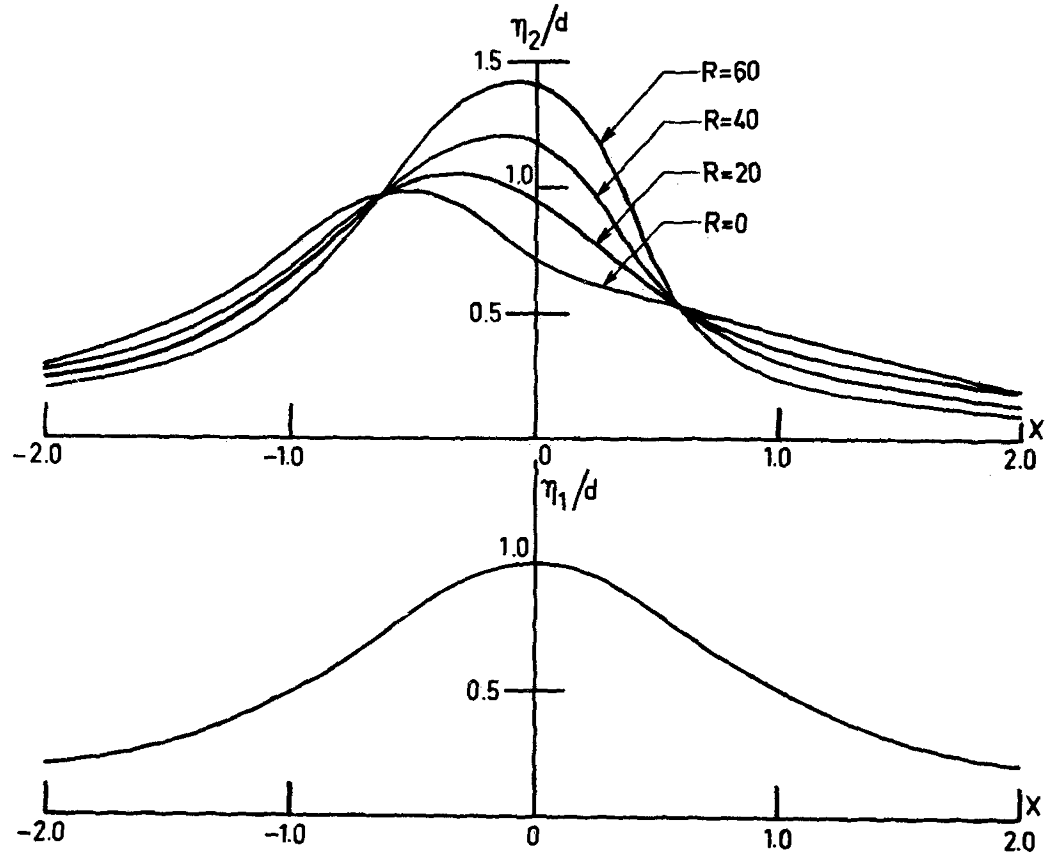

Figure 5. The shape of the surface wave for several values of the Reynolds number R. The perturbation wave is given below. The vertical scale is exaggerated.

the perturbation $\eta_1(x)/d$ displayed in the lower curve. The vertical scale is of course exaggerated since X is dimensionless. The parametric values $\epsilon = 0.25$ and $\cot\beta = 4$ were chosen to make the numerical work tolerable and at the same time to achieve a large phase difference between the perturbation and the free surface for $R = 0$. The surface is shown for $R = 0$, 20, 40, 60. The figure indicates that the wave becomes sharper as R increases, and for R greater than about 8 its amplitude exceeds that of the disturbance. The inertia effects also carry crest of the wave into phase with the disturbance.

The kinematic viscosity of glycerine at 20° C. is 6.798 cm^2/ sec. ([9], p. 7). We can easily compute the depth of liquid which corresponds to the Reynolds number $R = 40$ for $\cot \beta = 4$. From the definition of R, $h = (3R\nu^2/9 \sin \beta)^{1/3} = 2.86$ cm for glycerine in the case cited. This adds a quantitative view of figure 5.

REFERENCES

1. Berker, R.: Intégration des équations du mouvement d'un fluide visqueux incompressible. Handbuch der Physik, Band VIII/2, Springer, 1963.

2. Benjamin, T.B.: Wave formation in laminar flow down an inclined plane. J.Fluid Mech. **2**, pp. 554-574, 1957.

3. Wehausen, J.V.: Free surface flows in Research Frontiers in Fluid Dynamics, edited by R.J.Seeger and G.Temple, pp. 534-640, Interscience, New York, 1965.

4. Binnie, A.M.: Experiments on the onset of wave formation on a film of water flowing down a vertical plane. J.Fluid Mech. **2**, pp. 551-553, 1957.

5. Binnie, A.M.: Instability in slightly inclined water channels. J.Fluid Mech. **5**, pp. 561-570, 1959.

6. Erdélyi, A.: Tables of integral transforms, Vol. 1. McGraw-Hill, New York, 1954.

7. Jones, D.S.: Generalised functions. McGraw-Hill, London, 1966.

8. Copson, E.T.: Asymptotic expansions. Cambridge University Press, 1965.

9. Schlichting, H.: Boundary Layer Theory. McGraw-Hill, New York, 1960.

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